

A rank-based approach to Zipf's law

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Abstract

An Atlas model is a rank-based system of continuous semimartingales for which the steady-state values of the processes follow a power law, or Pareto distribution. For a power law, the log-log plot of these steady-state values versus rank is a straight line. Zipf's law is a power law for which the slope of this line is -1 . In this note, rank-based conditions are found under which an Atlas model will follow Zipf's law. An advantage of this rank-based approach is that it provides information about the dynamics of systems that result in Zipf's law.

Introduction

A family of random variables follows a *power law*, or *Pareto distribution*, if the log-log plot of their values versus rank forms (approximately) a straight line. The random variables follow *Zipf's law* if the slope of this line is -1 . Newman (2006) and Gabaix (2009) both present surveys of many different power laws observed in the real world. A characterization of conditions that result in Zipf's law for the population of cities is presented in Gabaix (1999), and this characterization is based on the idea that under a stable distribution the expected change in the population of each individual city is zero, at least when the city is away from a reflecting lower barrier.

In the setting of Atlas models and other systems of rank-based continuous semimartingales (see Fernholz (2002)), we examine the conditions that give rise to Zipf's law and consider several generalizations that are common in the real world. We shall find that this new setting is natural for an understanding of Zipf's law and provides insight into the dynamics involved.

Atlas models

An Atlas model is a family of positive continuous semimartingales X_1, \dots, X_n , with $n \geq 2$, that satisfy

$$d \log X_i(t) = (\gamma - g + ng \mathbb{1}_{\{r_t(i)=n\}}) dt + \sigma dW_i(t), \quad (1)$$

for $i = 1, \dots, n$, where γ is a constant, g and σ are positive constants, $r_t(i)$ is the rank of $X_i(t)$ (with ties resolved lexicographically), and W_1, \dots, W_n is an n -dimensional Brownian motion with the Brownian filtration \mathcal{F}_t (see Fernholz (2002), Banner et al. (2005), and Fernholz and Karatzas (2009)). The processes X_i might represent, for example, the wealth of households, the capitalizations of companies, or the population of cities. Let $X_{(1)} \geq \dots \geq X_{(n)}$ represent the ranked processes X_1, \dots, X_n , so $X_{(r_t(i))}(t) = X_i(t)$. We can define the *total value process* X by

$$X(t) \triangleq X_1(t) + \dots + X_n(t),$$

and the *weight processes* θ_i and the *ranked weight processes* $\theta_{(k)}$ by

$$\theta_i(t) \triangleq X_i(t)/X(t) \quad \text{and} \quad \theta_{(k)}(t) \triangleq X_{(k)}(t)/X(t), \quad \text{for } i, k = 1, \dots, n.$$

The term $ng \mathbb{1}_{\{r_t(i)=n\}}$ in (1) is a device that stabilizes the model by driving the “Atlas” process $X_{(n)}$ upward at a rate that counteracts the general downward drift of $-g$. The Atlas process can be thought of as the “birth and death” of processes in the lowest ranks, as is common in the firm size and income distribution literatures in economics (see Luttmer (2011) and Gabaix et al. (2015)). It can also be thought of as a

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proxy for a system that extends infinitely downward, as in the infinite models of Pal and Pitman (2008) and Chatterjee and Pal (2009).

The parameter γ in (1) represents the growth rate of the entire system. Since here we are interested in relative behavior under steady-state conditions, we can assume that $\gamma = 0$, and we shall do so from here on. In this case, definition (1) reduces to

$$d \log X_i(t) = (-g + ng \mathbb{1}_{\{r_t(i)=n\}}) dt + \sigma dW_i(t), \quad (2)$$

for $i = 1, \dots, n$. With this defining equation, the asymptotic growth rate of each of the X_i will be zero, so

$$\lim_{t \rightarrow \infty} t^{-1} \log X_i(t) = 0, \quad \text{a.s.},$$

for $i = 1, \dots, n$ (see, e.g., Fernholz (2002), Banner et al. (2005), or Fernholz and Karatzas (2009)).

By Itô's rule, it follows from (2) that

$$dX_i(t) = \left(-g + \frac{\sigma^2}{2} + ng \mathbb{1}_{\{r_t(i)=n\}} \right) X_i(t) dt + \sigma X_i(t) dW_i(t), \quad \text{a.s.}, \quad (3)$$

for $i = 1, \dots, n$. From this we see that while the asymptotic growth rate of the system (2) is zero, the local behavior of the individual processes X_i is more complicated.

For the model (2), when the gap processes $\log X_{(k)} - \log X_{(k+1)}$ are in their steady-state distribution, these gaps are exponentially distributed with

$$\mathbb{E}[\log X_{(k)}(t) - \log X_{(k+1)}(t)] = \mathbb{E}[\log \theta_{(k)}(t) - \log \theta_{(k+1)}(t)] = \frac{\sigma^2}{2kg}, \quad (4)$$

for $k = 1, \dots, n-1$ (see Banner et al. (2005)). It follows from (4) that

$$\frac{\mathbb{E}[\log \theta_{(k)}(t) - \log \theta_{(k+1)}(t)]}{\log k - \log(k+1)} = \frac{\sigma^2}{2kg(\log k - \log(k+1))} \cong -\frac{\sigma^2}{2g}, \quad \text{a.s.}, \quad (5)$$

for $k = 1, \dots, n-1$, and the log-log plot of $\theta_{(1)}(t), \dots, \theta_{(n)}(t)$ versus rank, which is called the *distribution curve* of the model, is approximately a straight line with (log-log) slope $-\sigma^2/2g$. Therefore, we have approximately

$$\theta_{(k)}(t) \propto k^{-\sigma^2/2g}, \quad \text{a.s.}, \quad (6)$$

for $k = 1, \dots, n$, and we say that the Atlas model (2) has a *Pareto distribution* with *parameter* λ , where

$$\lambda = \frac{\sigma^2}{2g}.$$

Zipfian Atlas models

Zipf's law is a Pareto distribution with parameter $\lambda = 1$, so the ranked weights in (6) will satisfy, approximately,

$$\theta_{(k)}(t) \propto k^{-1}, \quad \text{a.s.}, \quad (7)$$

for $k = 1, \dots, n$, and the distribution curve will be a straight line with slope -1 (see Gabaix (1999) or Gabaix (2009)). Gabaix (1999) constructs a simple economic model that yields a Pareto distribution for city size and then normalizes city populations so that the expected population of each city is constant, at least when a city is away from a lower reflecting barrier. This requirement of constant expected population results in Zipf's law.

For an Atlas model, when $r_t(i) < n$, equation (3) becomes

$$dX_i(t) = \left(-g + \frac{\sigma^2}{2} \right) X_i(t) dt + \sigma X_i(t) dW_i(t), \quad \text{a.s.}, \quad (8)$$

and

$$\mathbb{E}[dX_i(t)|\mathcal{F}_t, r_t(i) < n] = 0, \quad \text{a.s.}, \quad (9)$$

if

$$\left(\frac{\sigma^2}{2g} - 1\right)X_i(t) = 0, \quad \text{a.s.} \quad (10)$$

Since $X_i(t) > 0$, a.s., the necessary and sufficient condition for this to hold is that $\sigma^2/2g = 1$. This condition is equivalent to the requirement that (8) be a martingale for $r_t(i) < n$ (see also Bruggeman (2016), Section 3.6).

Definition 1. An Atlas model of the form (2) is *Zipfian* if $\sigma^2/2g = 1$.

For a Zipfian Atlas model,

$$\lambda = \frac{\sigma^2}{2g} = 1,$$

so it follows from (6) that (7) holds, which is exactly Zipf's law. We shall see in the next section that although a Zipfian Atlas model is distributed according to Zipf's law, a model that follows Zipf's law is not necessarily Zipfian.

Weakly Zipfian Atlas models

Empirically, for observed Zipf-like distributions it is not unusual for the distribution curve to be concave with the slope of the tangent flatter than -1 for the higher ranks and steeper than -1 for the lower ranks. Figure I in Gabaix (1999), Figure 5.1 in Fernholz (2002), and Figure 11 in Fernholz and Koch (2016) document this tendency, respectively, for the population of U.S. cities, the total market capitalizations of U.S. stocks, and the assets of U.S. bank holding-companies after the 1990s. The changing slopes in these real-world phenomena could be the result of variances σ_k^2 that increase with rank, as is conjectured for city size by Gabaix (2009), and is documented for stock capitalizations in Figure 5.5 of Fernholz (2002).

In order to study the case of increasing variances, let us consider a generalized Atlas model with variances that depend on rank. For $n \geq 2$, let

$$d \log X_i(t) = (-g + ng \mathbb{1}_{\{r_t(i)=n\}})dt + \sigma_{r_t(i)} dW_i(t), \quad (11)$$

for $i = 1, \dots, n$, where g and $\sigma_1, \dots, \sigma_n$ are positive constants, $r_t(i)$ is the rank of $X_i(t)$, and W_1, \dots, W_n is an n -dimensional Brownian motion (see, e.g., Fernholz (2002) or Banner et al. (2005)). Here all the ranks share a common reversion rate g , but each rank k has its own variance rate σ_k^2 . Banner et al. (2005) show that for a system of this form, if the gap processes $\log X_{(k)} - \log X_{(k+1)}$ are in their steady-state distribution, then

$$\mathbb{E}[\log \theta_{(k)}(t) - \log \theta_{(k+1)}(t)] = \frac{\sigma_k^2 + \sigma_{k+1}^2}{4kg},$$

for $k = 1, \dots, n-1$, so the tangent to the distribution curve between rank k and rank $k+1$ has log-log slope of

$$\frac{\mathbb{E}[\log \theta_{(k)}(t) - \log \theta_{(k+1)}(t)]}{\log k - \log(k+1)} = \frac{\sigma_k^2 + \sigma_{k+1}^2}{4kg(\log k - \log(k+1))} \cong -\frac{\sigma_k^2 + \sigma_{k+1}^2}{4g}. \quad (12)$$

Let us note that this slope is consistent with the slope (5) for the standard Atlas model (2).

From (12) we can construct an example of a generalized Atlas model for which Zipf's law holds, but the model is not Zipfian. For an even number n and $g > 0$, let

$$\sigma_{2j}^2 = g \quad \text{and} \quad \sigma_{2j+1}^2 = 3g,$$

for $j = 1, \dots, n/2$. For these values, by (12),

$$\frac{\mathbb{E}[\log \theta_{(k)}(t) - \log \theta_{(k+1)}(t)]}{\log k - \log(k+1)} \cong -\frac{\sigma_k^2 + \sigma_{k+1}^2}{4g} = -1,$$

for $k = 1, \dots, n-1$, so the log-log slope of the tangent to the distribution curve is -1 for all ranks, which means that Zipf's law holds. However, we see from (13) that $\sigma_{r_t(i)}^2/2g \neq 1$ for any X_i . Hence, Zipf's law holds for a Zipfian Atlas model, but a generalized Atlas model for which Zipf's law holds need not be Zipfian.

For a generalized Atlas model (11), equation (3) becomes

$$dX_i(t) = \left(-g + \frac{\sigma_{r_t(i)}^2}{2} + ng\mathbb{1}_{\{r_t(i)=n\}} \right) X_i(t) dt + \sigma_{r_t(i)} X_i(t) dW_i(t), \quad \text{a.s.}, \quad (13)$$

for $i = 1, \dots, n$. Let us assume that the gap processes $\log X_{(k)} - \log X_{(k+1)}$ are in their steady-state distribution. For variable σ_k^2 , we cannot expect that $\sigma_{r_t(i)}^2/2g = 1$ for all i with $r_t(i) < n$, so this model cannot be Zipfian. Instead, a more general definition is needed, so let us consider the *adjusted total value process* \tilde{X} defined by

$$d\tilde{X}(t) \triangleq dX(t) - ngX_{(n)}(t) dt. \quad (14)$$

We would like to impose conditions such that

$$\mathbb{E}[d\tilde{X}(t)|\mathcal{F}_t] = 0, \quad \text{a.s.}, \quad (15)$$

which is a natural generalization of the expected change in X_i when $r_t(i) < n$ in (9). We see from (13) that

$$\begin{aligned} d\tilde{X}(t) &= \sum_{i=1}^n dX_i(t) - ngX_{(n)}(t) dt \\ &= \sum_{i=1}^n \left(-g + \frac{\sigma_{r_t(i)}^2}{2} \right) X_i(t) dt + \sum_{i=1}^n \sigma_{r_t(i)} X_i(t) dW_i(t), \quad \text{a.s.} \end{aligned}$$

Hence, condition (15) implies that

$$\sum_{i=1}^n \left(\frac{\sigma_{r_t(i)}^2}{2g} - 1 \right) X_i(t) = \sum_{k=1}^n \left(\frac{\sigma_k^2}{2g} - 1 \right) X_{(k)}(t) = 0, \quad \text{a.s.}, \quad (16)$$

which is a natural generalization of (10). Since $X(t) > 0$, a.s., we can divide by it in (16) and take the expectation, which gives us the condition

$$\sum_{k=1}^n \left(\frac{\sigma_k^2}{2g} - 1 \right) \mathbb{E}[\theta_{(k)}(t)] = 0, \quad (17)$$

where the expected weights satisfy $\mathbb{E}[\theta_{(1)}(t)] > \dots > \mathbb{E}[\theta_{(n)}(t)] > 0$, and $\mathbb{E}[\theta_{(1)}(t)] + \dots + \mathbb{E}[\theta_{(n)}(t)] = 1$.

Definition 2. A generalized Atlas model of the form (11) is *weakly Zipfian* if (17) holds.

Now suppose that a generalized Atlas model is weakly Zipfian and that the variances $\sigma_1^2 < \dots < \sigma_n^2$ are increasing with rank. In this case the values of $\sigma_k^2/2g$ for the larger weights $\theta_{(k)}(t)$, i.e., for smaller k , will be less than one, and the values of $\sigma_k^2/2g$ for the smaller weights $\theta_{(k)}(t)$, i.e., for larger k , will be greater than one. The same will be true for $(\sigma_k^2 + \sigma_{k+1}^2)/4g$, so the distribution curve will be concave, with the slope of the tangent flatter than -1 for the higher ranks and steeper than -1 for the lower ranks. Indeed, as we noted above, concavity of this nature is consistent with many empirical distribution curves: see, e.g., Gabaix (1999), Figure I, or Fernholz (2002), Figure 5.1.

Other Zipfian systems

In the models (2) and (11) all the ranks share a common reversion rate g . However, Atlas models can be further generalized to *first-order models*, which are systems of the form

$$d \log X_i(t) = g_{r_t(i)} dt + \sigma_{r_t(i)} dW_i(t), \quad (18)$$

for $i = 1, \dots, n$, where $\sigma_1, \dots, \sigma_n$ are positive constants, $r_t(i)$ is the rank of $X_i(t)$, W_1, \dots, W_n is an n -dimensional Brownian motion, and g_1, \dots, g_n are constants such that

$$g_1 + \dots + g_n = 0, \quad \text{and} \quad g_1 + \dots + g_m < 0 \text{ for } m < n. \quad (19)$$

(see, e.g., Fernholz (2002) or Banner et al. (2005)). For these models,

$$\mathbb{E}[\log \theta_{(k)}(t) - \log \theta_{(k+1)}(t)] = \frac{\sigma_k^2 + \sigma_{k+1}^2}{-4(g_1 + \dots + g_k)},$$

for $k = 1, \dots, n-1$, so the tangent to the distribution curve between rank k and rank $k+1$ will have a log-log slope of

$$\frac{\mathbb{E}[\log \theta_{(k)}(t) - \log \theta_{(k+1)}(t)]}{\log k - \log(k+1)} = \frac{\sigma_k^2 + \sigma_{k+1}^2}{-4(g_1 + \dots + g_k)(\log k - \log(k+1))} \cong \frac{k(\sigma_k^2 + \sigma_{k+1}^2)}{4(g_1 + \dots + g_k)}. \quad (20)$$

Note that this slope is consistent with the slopes (5) and (12) for the previous more restrictive models (2) and (11). From (20) we see that the generalized model (18) can be parameterized to fit an arbitrary strictly decreasing distribution curve. Nevertheless, these results do not appear to suggest any obvious further generalizations (at least not to the authors).

There is a further generalization of these models that might be worthy of mention since it would accommodate a generalization of Zipf's law that appears in Gabaix (1999), Section III.2. In this more general case, we would consider a version of (18) where the X_i depend on parameters based on both rank and *index* or *name*. These *hybrid Atlas models* were introduced by Ichiba et al. (2011), however, parameter estimation for these models has not been completely resolved (as far as the authors know; see Fernholz et al. (2012)).

Example: the “size effect” for stocks

It was observed by Banz (1981) that stocks of U.S. companies with smaller capitalizations can be expected to have higher returns on average than stocks of U.S. companies with larger capitalizations. The explanation for this “anomaly” was considered to be the higher risk involved in holding smaller stocks (see Fama and French (1993)). Here we present a simple structural explanation based on the weak version of Zipf's law.

Suppose that the processes X_i in (11) represent the capitalizations of U.S. companies. It follows from (13) that the *relative return* of the stock X_i at time t is

$$\frac{dX_i(t)}{X_i(t)} = \left(-g + \frac{\sigma_{r_t(i)}^2}{2} + ng \mathbb{1}_{\{r_t(i)=n\}} \right) dt + \sigma_{r_t(i)} dW_i(t), \quad \text{a.s.}, \quad (21)$$

and similarly for the more restrictive configuration (8), where σ replaces $\sigma_{r_t(i)}$. Note that we are using relative return, since we removed the overall growth γ from the general model (1). For simplicity, we have ignored the payment of dividends or other distributions as a source of return since it was shown in Fernholz (1998) that the difference in these payments between large and small U.S. stocks had little influence on the observed size effect.

For a Zipfian model we have $\sigma_{r_t(i)}^2/2g = \sigma^2/2g = 1$, so it follows from (21) that

$$\mathbb{E} \left[\frac{dX_i(t)}{X_i(t)} \middle| r_t(i) = k \right] = \left(\frac{\sigma^2}{2g} - 1 \right) g dt = 0, \quad \text{a.s.}, \quad (22)$$

for $k = 1, \dots, n-1$. Hence, the expected relative return of each stock above the bottom rank is zero. However, for a Zipfian model the distribution curve will be linear, and we know that the distribution curve for stock capitalizations is concave rather than linear (see, e.g., Ijiri and Simon (1974) or Fernholz (2002), Figure 5.1). Therefore, the model can be at most weakly Zipfian.

Suppose the model is weakly Zipfian with increasing variances $\sigma_1^2 < \dots < \sigma_n^2$, which is consistent with a concave distribution curve. Instead of (22), we now have

$$\mathbb{E} \left[\frac{dX_i(t)}{X_i(t)} \middle| r_t(i) = k \right] = \left(\frac{\sigma_k^2}{2g} - 1 \right) g dt, \quad \text{a.s.}, \quad (23)$$

for $k = 1, \dots, n - 1$. Condition (17) and the increasing variances $\sigma_1^2 < \dots < \sigma_n^2$ imply that a large stock X_i will have lower variance, so $\sigma_k^2/2g < 1$, and the conditional expectation in (23) will be negative, while a small stock X_i will have higher variance, so $\sigma_k^2/2g > 1$, and the conditional expectation in (23) will be positive. Hence, the expected return on small stocks will be greater than the expected return on large stocks, and this provides a natural structural explanation for the size effect.

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